

# Axially Symmetric Solutions for SU(2) Yang-Mills Theory

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## Abstract

By casting the Yang-Mills-Higgs equations of an SU(2) theory in the form of the Ernst equations of general relativity, it is shown how the known exact solutions of general relativity can be used to give similar solutions for Yang-Mills theory. Thus all the known exact solutions of general relativity with axial symmetry (*e.g.* the Kerr metric, the Tomimatsu-Sato metric) have Yang-Mills equivalents. In this paper we only examine in detail the Kerr-like solution. It will be seen that this solution has surfaces where the gauge and scalar fields become infinite, which correspond to the infinite redshift surfaces of the normal Kerr solution. It is speculated that this feature may be connected with the confinement mechanism since any particle which carries an SU(2) color charge would tend to become trapped once it passes these surfaces. Unlike the Kerr solution, our solution apparently does not have any intrinsic angular momentum, but rather appears to give the non-Abelian field configuration associated with concentric shells of color charge.

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## I. INTRODUCTION

Recently, using the long known connection between general relativity and Yang-Mills theories [1], we found exact Schwarzschild-like solutions for Yang-Mills theories [2] [3], which were conjectured to have a possible connection with the confinement problem. Our results were conceptually similar to several other recent papers [4] which discussed related solutions. A natural question which arises from this is if there are other exact solutions of general relativity which have corresponding exact Yang-Mills solutions. Of particular interest is the Kerr solution which has an intrinsic angular momentum. In this paper we show that there are such solutions by considering an  $SU(2)$  gauge theory coupled to a scalar field in the adjoint representation. The solutions are found by first transforming the Yang-Mills field equations into the Ernst equations [5] of general relativity, and then applying the form of the general relativistic solution in terms of the gauge and scalar fields. Even though we specialize in this paper to the Kerr-like solution, it is in principle possible to use this same procedure to map over any axially symmetric solutions to Einstein's equations into an equivalent Yang-Mills solution. However, as we will show, even the Kerr-like solution has a very complex structure which makes it difficult to deal with. In addition it may be possible to reverse the above method and use known exact solution of Yang-Mills theory (*e.g.* the BPS dyon solution [6] [7] and multimonopole solutions [8]) to write down undiscovered solutions to Einstein's equations. One of the most interesting features of our previous Schwarzschild-like solution was the existence of a spherical shell surrounding the origin, on which the gauge and scalar fields became infinite, implying the presence of a spherical distribution of color charge. The Yang-Mills Kerr solution, in contrast, has two concentric shells of  $SU(2)$  charge (these shells are the equivalent of the infinite redshift surfaces of the normal Kerr solution). These infinite field surfaces might yield a possible mechanism for confinement, since just as general relativistic black holes permanently trap any object that carries gravitational "charge" (*i.e.* mass-energy), and which moves inside the event horizon, so the Yang-Mills version of these solutions may confine any particle which carries the gauge charge and crosses the color

event horizons. Actually, what we call the color event horizon of our solution, corresponds to the infinite redshift surfaces rather than the true event horizons of the normal Kerr metric. The reason for calling these surfaces the color event horizons is that the physical quantities of our theory (the gauge and scalar fields) develop real singularities on these surfaces. These infinite values of the fields imply, at least classically, that a particle carrying a color charge would either be strongly repelled or strongly attracted by these surfaces. For general relativity the corresponding singular surfaces are coordinate singularities which arise because of the particular coordinates that one chooses. This can best be seen for the normal Schwarzschild solution where, by transforming to Kruskal coordinates, one can eliminate the singularity in the metric at the Schwarzschild radius. In addition to this difference in the nature of the singularities, it is shown that while the regular Kerr solution has some angular momentum, our Yang-Mills version does not. These differences arise because the symmetries of general relativity are space-time symmetries, while the Yang-Mills symmetries are internal Lie symmetries.

## II. THE KERR-LIKE SOLUTION

The theory which we consider is an  $SU(2)$  gauge field which is coupled to a scalar field in the adjoint representation, which has no self-interaction or mass terms. The Lagrangian for this theory is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a + \frac{1}{2}D_\mu(\phi^a)D^\mu(\phi^a) \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc}W_\mu^b W_\nu^c \quad (2)$$

and

$$D_\mu\phi^a = \partial_\mu\phi^a + g\epsilon^{abc}W_\mu^b\phi^c \quad (3)$$

To obtain the Bogomolny field equations associated with this theory one finds the field configuration which produces an extremum in the canonical Hamiltonian

$$\mathcal{H} = \int d^3x \left[ \frac{1}{4} F_{ij}^a F^{aij} - \frac{1}{2} F_{0i}^a F^{a0i} + \frac{1}{2} D_i \phi^a D^i \phi^a - \frac{1}{2} D_0 \phi^a D^0 \phi^a \right] \quad (4)$$

We now introduce an explicit scale factor for the scalar field (*i.e.*  $\phi^a \rightarrow A\phi^a$ ) so that we can study the special case with no scalar field by simply taking  $A = 0$ . Next we require that all the fields are time independent and that the time components of the gauge fields are proportional to the scalar fields (*i.e.*  $W_0^a = C\phi^a$ , where  $\phi^a$  is the rescaled field). Thus  $W_0^a$  acts like an additional Higgs field except that its kinetic term appears with the opposite sign in Eq. (4). Using all these conditions and the antisymmetry of  $\epsilon^{abc}$  we find that  $D_0 \phi^a = 0$  and  $F_{0i}^a = C(D_i \phi^a)$ , so that the Hamiltonian becomes

$$\mathcal{H} = \int d^3x \left[ \frac{1}{4} (F_{ij}^a - \epsilon_{ijk} \sqrt{A^2 - C^2} D^k \phi^a) (F^{aij} - \epsilon_{ijl} \sqrt{A^2 - C^2} D^l \phi^a) + \frac{1}{2} \epsilon_{ijk} \sqrt{A^2 - C^2} F^{aij} D^k \phi^a \right] \quad (5)$$

Using the relationship  $\epsilon_{ijk} F^{aij} D^k \phi^a = \partial^i (\epsilon_{ijk} F^{ajk} \phi^a)$  the last term in Eq. (5) can be turned into a surface intergral, which in the usual development [6] is proportional to the magnetic charge carried by the fields due to the topology of the Higgs field at infinity [9]. The lower limit of the above Hamiltonian can be found by requiring

$$F_{ij}^a = \sqrt{A^2 - C^2} \epsilon_{ijk} D^k \phi^a \quad (6)$$

which are the usual Bogomolny field equations, with the presense of the scalar and time component of the gauge fields explicitly displayed through the constants  $A$  and  $C$ . This will make it easier to examine several special cases later on.

Several exact solutions to the field equations of this theory have been found which possess spherical symmetry : the Bogomolny-Prasad-Sommerfield dyon solution [6] [7], and more recently a Schwarzschild-like solution [2]. In this paper we are looking for an axially symmetric solution. Several authors [10] [11] have already given an axially symmetric ansatz for the gauge and scalar fields

$$\begin{aligned}
\Phi^a &= (0, \phi_1, \phi_2) & W_\phi^a &= -(0, \eta_1, \eta_2) \\
W_z^a &= (W_1, 0, 0) & W_\rho^a &= -(W_2, 0, 0)
\end{aligned} \tag{7}$$

where  $\varphi, z, \rho$  are the usual polar coordinates and  $\phi_i, \eta_i, W_i$  are functions of  $\rho, z$  only. With this ansatz the the Bogomolny equations of Eq. (6) become [10]

$$\begin{aligned}
\rho\sqrt{A^2 - C^2}(\partial_\rho\phi_1 - W_2\phi_2) &= -(\partial_z\eta_1 - W_1\eta_2) \\
\rho\sqrt{A^2 - C^2}(\partial_\rho\phi_2 + W_2\phi_1) &= -(\partial_z\eta_2 + W_1\eta_1) \\
\rho(\partial_\rho W_1 - \partial_z W_2) &= \sqrt{A^2 - C^2}(\phi_1\eta_2 - \phi_2\eta_1) \\
\rho\sqrt{A^2 - C^2}(\partial_z\phi_1 - W_1\phi_2) &= (\partial_\rho\eta_1 - W_2\eta_2) \\
\rho\sqrt{A^2 - C^2}(\partial_z\phi_2 + W_1\phi_1) &= (\partial_\rho\eta_2 + W_2\eta_1)
\end{aligned} \tag{8}$$

If one defines two new functions,  $f(\rho, z)$  and  $\psi(\rho, z)$ , such that the fields,  $\eta_i, \phi_i$  and  $W_i$  are written as

$$\begin{aligned}
\phi_1 &= -W_1 = \frac{1}{f} \frac{\partial\psi}{\partial z} \\
\eta_1 &= \rho W_2 = -\frac{\rho}{f} \frac{\partial\psi}{\partial\rho} \\
\phi_2 &= -\frac{1}{f} \frac{\partial f}{\partial z} \\
\eta_2 &= \frac{\rho}{f} \frac{\partial f}{\partial\rho}
\end{aligned} \tag{9}$$

then the Bogomolny equations of Eq. (8) become [8]

$$Re(\varepsilon)\nabla^2\varepsilon = \nabla\varepsilon \cdot \nabla\varepsilon \tag{10}$$

where  $\varepsilon = f + i\psi$ , and  $\nabla^2$  and  $\nabla$  are the Laplacian and gradient in cylindrical coordinates.

Eq. (10) is the Ernst equation [5] of general relativity. This form of the Bogomolny equations has been used to find exact, nonsingular, multimonopole solutions for the fields through the use of the Bäcklund transformations of Harrison [12]. Since the axial Bogomolny equations can be written in the form of the Ernst equations, it should be possible to use the known exact solutions of general relativity to find exact solutions for SU(2) Yang-Mills-Higgs theory. That this link between the general relativistic solutions and their Yang-Mills

counterparts has not been exploited before, can perhaps be attributed to the singularities which exist in these solutions. Here it is conjectured that these singularities might actually be a desired feature in that they may provide a confinement mechanism for non-Abelian gauge theories. Previously [2] [3], using a different approach we have found exact Schwarzschild-like solutions for SU(2) and SU(N) Yang-Mills-Higgs theories. Here we use the well known Kerr solution, written in terms of variables of the Ernst equation, to give an equivalent solution for the Yang-Mills theory. There are actually several other exact, axially symmetric solutions in general relativity which could be mapped over into Yang-Mills theory (*e.g.* the Tomimatsu-Sato metric [13] and the NUT-Taub metric [14]). However the Kerr metric is the simplest axially symmetric solution, and is of physical interest since it gives the exterior gravitational field for a central mass with angular momentum. However our axial Yang-Mills solution apparently does not possess any angular momentum, but rather seems to represent the non-Abelian field configuration due to two concentric shells of SU(2) charge. Thus although the Kerr-like solution is found using the general relativistic solution, they appear to have some different physical characteristics.

To find the Kerr solution from the Ernst equation one first introduces the complex potential  $\zeta$  such that

$$\varepsilon = f + i\psi = \frac{\zeta - 1}{\zeta + 1} \quad (11)$$

so that  $f$  and  $\psi$  are the real and imaginary parts, respectively, of  $(\zeta - 1)/(\zeta + 1)$ . Substituting this expression for  $\varepsilon$  into, Eq. (10), the Ernst equation becomes

$$(\zeta\bar{\zeta} - 1)\nabla^2\zeta = 2\bar{\zeta}\nabla\zeta \cdot \nabla\zeta \quad (12)$$

where  $\bar{\zeta}$  is the complex conjugate of  $\zeta$ . For this form of the equations the Kerr solution is most easily found using prolate spheroidal coordinates [5], which can be written in terms of the cylindrical coordinates,  $\rho$  and  $z$ , as

$$\begin{aligned} x &= \frac{1}{2k} \left[ \sqrt{(z+k)^2 + \rho^2} + \sqrt{(z-k)^2 + \rho^2} \right] \\ y &= \frac{1}{2k} \left[ \sqrt{(z+k)^2 + \rho^2} - \sqrt{(z-k)^2 + \rho^2} \right] \end{aligned} \quad (13)$$

where the inverse transformation is given by

$$\begin{aligned}\rho &= k\sqrt{x^2 - 1}\sqrt{1 - y^2} \\ z &= kxy\end{aligned}\tag{14}$$

where  $k, p$  and  $q$  are arbitrary constants. In these prolate spheroidal coordinates the gradient and Laplacian become

$$\begin{aligned}\nabla &= \frac{k}{\sqrt{x^2 - y^2}} \left[ \hat{\mathbf{x}}\sqrt{x^2 - 1}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\sqrt{1 - y^2}\frac{\partial}{\partial y} \right] \\ \nabla^2 &= \frac{k^2}{x^2 - y^2} \left[ \frac{\partial}{\partial x}(x^2 - 1)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(1 - y^2)\frac{\partial}{\partial y} \right]\end{aligned}\tag{15}$$

where  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are unit vectors. Using these expressions it is easy to see that a solution to the second form of the Ernst equation, Eq. (12) is

$$\zeta = px - iqy\tag{16}$$

where the constants  $p, q$  must satisfy the condition  $p^2 + q^2 = 1$ . The above solution can be transformed into the standard form of the Kerr solution by doing a transformation from the prolate spheroidal coordinates to Boyer-Lindquist coordinates [5]. The special case when  $q = 0, p = 1$  and  $\zeta = x$  gives (after a transformation to Schwarzschild coordinates) the usual Schwarzschild metric for general relativity. However using the solution  $\zeta = x$  in the Yang-Mills case to write down the expressions for the gauge and scalar fields, we find that we do not recover our previous Schwarzschild-like solution for  $SU(2)$ , but obtain a different solution. The reason for this lies in the fact that the ansätze we used in each case were different, and in the previous work we found our solution directly from the Euler-Lagrange field equations, while here we employed the Bogomolny formalism. A field configuration that satisfies the Bogomolny equations will also satisfy the Euler-Lagrange equations, but the reverse is not necessarily true. Nevertheless  $\zeta = x$  does give a solution to the Yang-Mills-Higgs equations, which is a special case of the general solution (*i.e.*  $p, q \neq 0$ ) that we are considering. In general relativity  $p, q$  and  $k$  are related to the mass and angular momentum of the central mass which produces the gravitational field. Here  $p, q$  and  $k$  will be related to

the shape of the axially symmetric SU(2) charge configuration of our solution. In order to find expressions for the fields  $\phi_i, \eta_i, W_i$  it is first necessary to determine the functions  $f$  and  $\psi$ . From Eq. (11) we find that

$$\begin{aligned} f &= \frac{p^2x^2 + q^2y^2 - 1}{(px + 1)^2 + q^2y^2} \\ \psi &= \frac{-2qy}{(px + 1)^2 + q^2y^2} \end{aligned} \quad (17)$$

To get the gauge and scalar fields one simply inserts these expressions for  $f$  and  $\psi$  into Eq. (9). This is a straightforward but tedious procedure which yields very complicated expressions for the fields. The explicit expressions for the fields are

$$\begin{aligned} \phi_1 = -W_1 &= \frac{-2q[x(1-y^2)((px+1)^2 - q^2y^2) - 2py^2(px+1)(x^2-1)]}{k[(px+1)^2 + q^2y^2](p^2x^2 + q^2y^2 - 1)(x^2 - y^2)} \\ \eta_1 = \rho W_2 &= \frac{-2qy[(px+1)^2 - q^2y^2 + 2px(px+1)](x^2-1)(1-y^2)}{[(px+1)^2 + q^2y^2](x^2 - y^2)(p^2x^2 + q^2y^2 - 1)} \\ \phi_2 &= \frac{-2xy[p^2(x^2-1) + q^2(1-y^2)]}{k(x^2 - y^2)(p^2x^2 + q^2y^2 - 1)} + \frac{2y[p(px+1)(x^2-1) + q^2x(1-y^2)]}{k(x^2 - y^2)[(px+1)^2 + q^2y^2]} \\ \eta_2 &= \frac{2(x^2-1)(1-y^2)(p^2x^2 - q^2y^2)}{(x^2 - y^2)(p^2x^2 + q^2y^2 - 1)} - \frac{2(x^2-1)(1-y^2)[px(px+1) - q^2y^2]}{(x^2 - y^2)[(px+1)^2 + q^2y^2]} \end{aligned} \quad (18)$$

where the partial derivatives like,  $\partial\psi/\partial z$ , were determined using the chain rule (*e. g.*  $\partial_x\psi\partial_zx$ ) and Eq. (13). Notice that in general relativity the physical quantities one usually deals with are the components of the metric tensor. Here the physical quantities are the gauge fields which correspond to the Christoffel symbols in general relativity. This partly explains the complexity of the expressions in Eq. (18), since even in Boyer-Lindquist coordinates, the Christoffel coefficients for the Kerr metric are somewhat involved. If one wanted to have the expressions for the fields in terms of the original coordinates, it would be necessary to use Eq. (13) to replace  $x, y$  with  $\rho, z$ , making an already complicated expression even more intractable. However by looking at certain aspects of the expressions of the fields one can still make some interesting comments about this solution.



One feature that can be looked for are regions where the fields become singular. In analogy with Maxwell's equations, where singularities in the electromagnetic field indicate the presence of electric charge, we interpret these singularities as the location of color charge. The shape of these SU(2) charge distributions is much more involved than in electromagnetism. All of the fields from Eq. (18) have three similar terms in their denominators, so the fields can be made to approach infinity if any one of the three factors goes to zero. First, by setting  $p^2x^2 + q^2y^2 - 1 = 0$  it can be seen that all the fields become infinite. This factor is common to all the fields, since from Eq. (9) they all have a factor of  $f^{-1}$ . In cylindrical coordinates this condition becomes

$$|pq|(z^2 + \rho^2 - k^2) = \pm|q^2 - p^2|\rho k \quad (19)$$

By setting  $q = \cos\theta$  and  $p = \sin\theta$  we replace the two parameters  $p, q$  with one parameter, and Eq. (19) becomes

$$z^2 + \rho^2 - k^2 = \pm 2|\cot(2\theta)|\rho k \quad (20)$$

Solving the above condition for  $z$  as a function of  $\rho$  allows one to take a vertical slice through the two axially symmetric surfaces defined by Eq. (20). One needs only to look in the range,  $0^\circ \leq \theta \leq 45^\circ$  to cover all the possibilities. What one finds are two concentric surfaces which touch each other on the  $z$ -axis at  $\pm k$ . The outer surface is given by the positive solution to Eq. (20). It has a toroidal shape, without the central hole of a normal torus. The inner surface is given by the negative solution, and has an ellipsoidal shape which runs along the  $z$ -axis. In the special case when  $\theta = 45^\circ$  (*i.e.*  $p = q$ ) the two surfaces merge into a single sphere with a radius of  $k$ . Second, the fields can become infinite if  $x^2 - y^2 = 0$ . However this condition gives two points ( $\rho = 0, z = \pm k$ ) which are already included in the first condition of Eq. (19). Finally some of the fields become singular when  $y = 0$  and  $px + 1 = 0$  (since  $x$  is positive definite this condition only has a solution when  $p < 0$ ). The condition  $y = 0$  implies  $z = 0$  so that the singularity resides in the plane perpendicular to the  $z$ -axis, and then  $px + 1 = 0$  gives  $\rho = k|q|/|p| = k|\cot\theta|$ .

This corresponds to a ring singularity of radius  $k|\cot\theta|$  centered at the origin in the plane perpendicular to the  $z$ -axis. However this singularity, produced by the condition  $y = 0$  and  $px + 1 = 0$ , duplicates that from the condition of Eq. (20). Thus the singularity produced by the third term in the denominators of the gauge fields does not produce an independent singularity. The geometrical structure of the singular surfaces is different from that of the similar Schwarzschild-like solution. For the Schwarzschild solution we obtained a spherical shell singularity surrounding a point singularity at the origin. In the present case we find concentric toroidal and ellipsoidal surfaces on which the gauge and scalar fields become infinite, and with no apparent singularity in the interior of these surfaces.

The connection of both the Schwarzschild-like solution and the Kerr-like solution with confinement can be seen both classically and quantum mechanically. First, from a classical point of view, the non-Abelian fields become arbitrarily large as one approaches the surfaces defined by Eq. (19). Any particle which carries an SU(2) charge would either be strongly attracted or strongly repelled as it approached these surfaces of Eq. (19), depending on the relative “sign” of the SU(2) charge of the particle to that of the surface. If the force between the particle and surface were repulsive, the particle would never be able to enter the interior region. If the force were attractive then once the particle entered the interior region it would no longer be able to leave, thus becoming permanently confined. This behaviour in the attractive case is similar to the behaviour of particles in the vicinity of a general relativistic black hole. The repulsive case is different from that of general relativity since Yang-Mills theories are vector theories, which lead to both attractive and repulsive forces, whereas general relativity is a tensor theory which leads only to attractive forces. Quantum mechanically the above configuration is similar in character to some phenomenological bag models of hadron dynamics, where QCD bound state are modeled as free particles (*e.g.* quarks) inside a spherical hard wall potential. In the case of the Schwarzschild-like solution it has been shown [4] that if a scalar or spinor particle with SU(2) charge is placed in the potential of the gauge fields it will remain confined to the region inside the surfaces of infinite field strength. Our analytic Schwarzschild-like solution [2] is slightly different from those of

Ref. [4] in that our solution is singular at the origin. However since the surface singularity is a common feature of both solutions we expect that our solution will also permanently confine any color charged particle.

The exact expressions for the energy and angular momentum of the Kerr-like solution presented here are rather complicated due to the involved nature of the scalar and gauge fields (see Eq. (18)). Still some interesting general conclusions can be made about these quantities. To find the energy and angular momentum in the fields it is necessary to calculate the energy-momentum tensor of the Lagrangian of Eq. (1)

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g_{\mu\nu}} = F^{\mu\rho a} F_{\rho}^{\nu a} + D^{\mu} \phi^a D^{\nu} \phi^a + g^{\mu\nu} \mathcal{L} \quad (21)$$

The energy of the field configuration is then

$$E = \int d^3x T^{00} = \int d^3x \left[ \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} F_{0i}^a F^{a0i} + \frac{A^2}{2} D_i \phi^a D^i \phi^a + \frac{A^2}{2} D_0 \phi^a D^0 \phi^a \right] \quad (22)$$

This is equivalent to the Hamiltonian except the signs are all positive. Now using  $D^0 \phi^a = 0$ ,  $F_{0i}^a = C(D_i \phi^a)$  and  $F_{ij}^a = \sqrt{A^2 - C^2} \epsilon_{ijk} D^k \phi^a$  the energy becomes

$$E = A^2 \int d^3x D_i \phi^a D^i \phi^a \quad (23)$$

The constant  $A$  is the multiplicative factor that we put in front of the scalar fields in order that we could easily examine the case when there were no scalar fields by taking  $A = 0$ . From Eq. (23) it can be seen that for this special case the energy in the fields of the Kerr-like solution is zero. In addition when  $A = 0$  either the time components or the space components of the gauge fields are pure imaginary. These results were the same for the Schwarzschild Yang-Mills solution, and we believe that this calls into question the physical relevance of the pure gauge case even though it is mathematically a solution. Taking this view then requires the presence of a scalar field in order to get a physically “reasonable” solution. The angular momentum of the field configuration is given by

$$L = \int d^3x \epsilon_{ijk} x^j T^{0k} \quad (24)$$

Using the expression for  $T^{\mu\nu}$  from Eq.(21), the condition  $F_{0i}^a = C(D_i\phi^a)$  and the Bogomolny field equations, Eq. (6), we find

$$T^{0k} = \epsilon^{klm} C \sqrt{A^2 - C^2} (D_l \phi^a) (D_m \phi^a) \quad (25)$$

The antisymmetry of  $\epsilon^{klm}$  makes  $T^{0k} = 0$ , so that there is no angular momentum in this non-Abelian field configuration. This shows that while our Yang-Mills solution is similar in many ways to its general relativistic counterpart, there are some important distinctions. Part of the reason for this stems from the fact that the symmetries of general relativity are space-time symmetries, while those of Yang-Mills theories are internal Lie symmetries. These differences showed up even in the Yang-Mills Schwarzschild-like solution where the sphere singularity was a true singularity, while for general relativity the event horizon is a coordinate singularity, as can be seen by looking at the Schwarzschild solution in Kruskal coordinates.

### III. DISCUSSION AND CONCLUSION

Extending our previous work on the Schwarzschild-like solution for Yang-Mills-Higgs theories, we have written down the Yang-Mills equivalent of the Kerr solution. By writing the Yang-Mills field equations in the form of the Ernst equation [11] of general relativity [5] it is straightforward to use any known axially symmetric solution of general relativity to write down similar solutions in terms of the non-Abelian gauge fields. One disadvantage of these general relativistic inspired solutions is that they contain singularities in the fields, which lead to infinite field energies at the classical level. These singularities are of the same character as the singularities which are found in other classical field theory solutions such as the singularity at the origin in the normal Schwarzschild solution, the point singularity in the Wu-Yang solution for SU(2) [15], and the singularity at  $r = 0$  in the Coulomb potential in electromagnetism. Since our solutions are classical field theory solutions it may be conjectured, as is the case with general relativity, that a proper quantum treatment of the

problem might modify these singularities. Fortunately, unlike the case of general relativity, there do exist methods for quantizing such classical solutions [16] [17]. In one sense however, these singularities (particularly the surface singularities) are a desirable feature in that they may yield a possible confinement mechanism for non-Abelian gauge theories, which would be analogous to the confinement mechanism of general relativistic black holes. Quantum mechanically it can be shown that these solutions lead to a simple form of confinement, by placing a particle with color charge into the potentials of our solutions and solving the relevant quantum problem. This was outlined in Ref. [4] where color charged test particles were placed in a Yang-Mills field configuration similar in some respects to ours. It was found that the wavefunction of the test particle was confined to the region around the origin, which is in accord with our initial findings using our Schwarzschild-like solution. While this quantum mechanical analysis is suggestive, it does have shortcomings. The chief one being that it ignores the interaction between the color field of the test particle and the field configuration in which it is placed. Since the field equations of Yang-Mills theories are non-linear one is not justified in using superposition, and the color field of the test particle could significantly alter the color field of the solution. This is similar to the problem in general relativity of the interaction of two black holes, which must be handled numerically. Nevertheless one could argue that two color charged particles could form a bound state in such a way that they move in an average field configuration which is given approximately by our solutions.

The very direct link, via the Ernst equations, between general relativity and Yang-Mills-Higgs theories can be used to map over any of the known axially symmetric solutions from general relativity into non-Abelian gauge theories. In this paper we examined in detail only the Kerr solution in the hope of finding a field configuration, which contained an internal angular momentum, so that quantizing this angular momentum to  $\hbar/2$  one would be able to have a fermion-like object from an initial theory with only gauge and scalar fields. Explicitly carrying out the calculation of the field angular momentum showed that even though the Kerr-like solution was axially symmetric, it did not carry any angular momentum in its fields.

This shows that not all the features of the general relativistic solution carry over into Yang-Mills theory. So far we have used this parallel between the two theories to find solutions for Yang-Mills theories from the known solutions of general relativity. An interesting exercise might be to see if some of the the exact solutions of Yang-Mills theory (*e.g.* the Prasad-Sommerfield solution or the multimonopole solutions [8]) could be used to give unknown exact solutions in general relativity.

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